

On a Class Number Formula for Real Quadratic Number Fields

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Abstract

For an even Dirichlet character ψ , we obtain a formula for $L(1, \psi)$ in terms of a sum of Dirichlet L -series evaluated at $s = 2$ and $s = 3$ and a rapidly convergent numerical series involving the central binomial coefficients. We then derive a class number formula for real quadratic number fields by taking $L(s, \psi)$ to be the quadratic L -series associated with these fields.

1 Introduction

In [1], acceleration formulæ are derived for Catalan's constant $L(2, \chi_4)$; (here χ_4 is the non-principal Dirichlet character of modulus 4). In some of these formulæ $L(2, \chi_4)$ is given as the sum of two terms: one involving a rapidly convergent series and the other involving the natural logarithm of a unit in the ring of integers of a finite abelian field extension of the rational number field \mathbb{Q} . The existence of the logarithmic terms suggested to the authors that these terms should somehow be related to the values of Dirichlet L -series at the argument $s = 1$. This leads to the general question of whether or not there exist relations between the value of L -series at $s = 1$ and values of L -series at integer arguments larger than 1.

The purpose of this note is to exhibit such a relation between values of L -series. For an even Dirichlet character ψ , we obtain a formula for $L(1, \psi)$ in terms of a sum of Dirichlet series evaluated at $s = 2$ and $s = 3$ and a convergent numerical series involving powers of twice special values of the sine function divided by $\binom{2n}{n} n^3$. See Theorem 1 below for a precise statement. (It is perhaps interesting to notice that not much is known about number theoretic properties of the values of the L -series on the right-hand side of the formula given in this theorem.) We then deduce a class number formula for real quadratic number fields by letting ψ be the quadratic character associated with a real quadratic number field; see Corollary 1. This class number formula seems new to us and is perhaps an interesting curiosity.

To derive our results, we employ a formula of Zucker [5] that expresses

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{\binom{2n}{n} n^3}, \quad |x| \leq 2, \quad (1)$$

in terms of periodic zeta functions. Proposition 1 below shows how periodic zeta functions may be expressed in terms of Dirichlet L -series. Thus, we can rewrite (1) in terms of L -series values, thereby obtaining our result.

2 Preliminaries

Let m be a positive integer. We denote the group of Dirichlet characters of modulus m by \hat{U}_m . The Dirichlet L -series associated with $\chi \in \hat{U}_m$ is

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \operatorname{Re}(s) > 1.$$

Similarly, for real β we define the *periodic zeta function* (a special case of the Lerch transcendent) by

$$\Phi(s, \beta) = \sum_{n=1}^{\infty} \frac{e^{2\pi i \beta n}}{n^s}, \quad \operatorname{Re}(s) > 1.$$

Let $\zeta_m = e^{2\pi i/m}$. Throughout, the sum over a complete set of residues modulo m is denoted by $\sum_{a \bmod m}$ and the sum over the positive integer divisors of m is denoted by $\sum_{d|m}$. Thus, Ramanujan's sum is

$$c_m(k) = \sum_{\substack{\nu \bmod m \\ (\nu, m)=1}} \zeta_m^{\nu k},$$

and likewise the Gaussian sum attached to χ is

$$\tau(\chi) = \sum_{\nu \bmod m} \chi(\nu) \zeta_m^{\nu}.$$

Also, $\overline{\chi}$ denotes the inverse—or equivalently, the complex conjugate—of the character χ . Finally, as customary, $\mu()$, $\varphi()$, and $\zeta()$ denote the Möbius, Euler totient, and Riemann zeta functions, respectively.

Our immediate goal is to represent periodic zeta functions in terms of L -series. It turns out to be easier to do the reverse first. The following result is well known, so we omit the proof.

Lemma 1 *Let m be a positive integer, let χ be a Dirichlet character of modulus m , and let $L(s, \chi)$ be the associated Dirichlet L -series. Then*

$$L(s, \chi) = \frac{1}{m} \sum_{a \bmod m} \chi(a) \sum_{b \bmod m} \zeta_m^{-ab} \Phi(s, b/m), \quad \operatorname{Re}(s) > 1.$$

Lemma 2 *Let a and m be positive integers. Then*

$$\frac{1}{\varphi(m)} \sum_{\chi \in \hat{U}_m} \chi(a) \tau(\bar{\chi}) L(s, \chi) = \frac{1}{m} \sum_{b \bmod m} \Phi(s, b/m) c_m(a-b), \quad \operatorname{Re}(s) > 1.$$

Proof. First recall that

$$\sum_{\chi \in \hat{U}_m} \bar{\chi}(c) \chi(a) = \begin{cases} \varphi(m) & \text{if } (ac, m) = 1 \text{ and } a \equiv c \bmod m, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that if $(c, m) = 1$, then

$$\frac{\varphi(m)}{m} \sum_{b \bmod m} \zeta_m^{-bc} \Phi(s, b/m) = \sum_{\chi \in \hat{U}_m} \chi(c) L(s, \chi); \quad (2)$$

for by Lemma 1,

$$\begin{aligned} \sum_{\chi \in \hat{U}_m} \bar{\chi}(c) L(s, \chi) &= \frac{1}{m} \sum_{a \bmod m} \sum_{b \bmod m} \zeta_m^{-ab} \Phi(s, b/m) \sum_{\chi \in \hat{U}_m} \bar{\chi}(c) \chi(a) \\ &= \frac{\varphi(m)}{m} \sum_{b \bmod m} \zeta_m^{-bc} \Phi(s, b/m). \end{aligned}$$

On the other hand, if $(c, m) > 1$, then clearly

$$\sum_{\chi \in \hat{U}_m} \bar{\chi}(c) L(s, \chi) = 0.$$

We now multiply equation (2) by ζ_m^{ac} with $(a, m) = 1$, and then sum over all c modulo m , obtaining

$$\begin{aligned} \frac{\varphi(m)}{m} \sum_{b \bmod m} \sum_{\substack{c \bmod m \\ (c, m) = 1}} \zeta_m^{(a-b)c} \Phi(s, b/m) &= \sum_{c \bmod m} \zeta_m^{ac} \sum_{\chi \in \hat{U}_m} \bar{\chi}(c) L(s, \chi) \\ &= \sum_{c \bmod m} \zeta_m^{ac} \sum_{\chi \in \hat{U}_m} \bar{\chi}(c) L(s, \chi) \\ &= \sum_{\chi \in \hat{U}_m} \sum_{c \bmod m} \bar{\chi}(c) \zeta_m^{ac} L(s, \chi) \\ &= \sum_{\chi \in \hat{U}_m} \chi(a) \tau(\bar{\chi}) L(s, \chi). \end{aligned}$$

Rewriting this latter equation in terms of Ramanujan sums completes the proof.

□

We now state the main proposition of this section.

Proposition 1 *Let a and m be coprime positive integers. Then*

$$m^s \Phi(s, a/m) = \sum_{d|m} \frac{d^s}{\varphi(d)} \sum_{\chi \in \hat{U}_d} \chi(a) \tau(\overline{\chi}) L(s, \chi), \quad \text{Re}(s) > 1.$$

Before proving Proposition 1, we state and prove two lemmata which are used in the proof of Proposition 1.

Lemma 3 *Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be multiplicative and such that for all positive integers m ,*

$$F(m) := \sum_{d|m} \mu^2(d) f(d)$$

is non-zero. Furthermore, let

$$g(m) := \sum_{d|m} \frac{\mu(d)}{F(d)}.$$

Then for all positive integers k and m such that k divides m ,

$$\sum_{\substack{d|m \\ k|d}} \mu^2(d) f(d) = F(m) \mu^2(k) g(k).$$

In particular,

$$\sum_{\substack{d|m \\ k|d}} \frac{\mu^2(d)}{\varphi(d)} = \frac{m}{\varphi(m)} \frac{\mu^2(k)}{k}.$$

Proof. First, let us define $F(x) = f(x) = 0$ if x is not an integer. Next, observe that $F(p^a) = F(p)$ for all positive primes p and positive integers a . We may write m as $\prod_p p^{a_p}$ and k as $\prod_p p^{b_p}$ where p ranges over all positive primes and a_p and b_p are non-negative integers with $b_p \leq a_p$. Since F is multiplicative, we have

$$\begin{aligned} \sum_{\substack{d|m \\ k|d}} \mu^2(d) f(d) &= \prod_{p|m} \left(\sum_{\nu_p=b_p}^{a_p} \mu^2(p^{\nu_p}) f(p^{\nu_p}) \right) \\ &= \prod_{p|m} (F(p^{a_p}) - F(p^{b_p-1})) \\ &= \prod_{p|m} F(p^{a_p}) \left(1 - \frac{F(p^{b_p-1})}{F(p^{a_p})} \right) \\ &= F(m) \prod_{p|m} \left(1 - \frac{F(p^{b_p-1})}{F(p^{a_p})} \right). \end{aligned} \tag{3}$$

Notice that the final product in (3) vanishes if any $b_p \geq 2$, for then $F(p^{b_p-1}) = F(p) = F(p^{a_p})$. Hence if k is not square-free, then the lemma is trivially true as both sides are equal to 0. Therefore, we may assume henceforth that k is square-free. Now if $b_p = 0$, then $1 - F(p^{b_p-1})/F(p^{a_p}) = 1$, and thus (under the assumption that k is square-free), we may restrict the final product in (3) to primes p for which $b_p = 1$. This yields

$$\begin{aligned} \sum_{\substack{d|m \\ k|d}} \mu^2(d)f(d) &= F(m) \prod_{p|k} \left(1 - \frac{1}{F(p^{a_p})}\right) = F(m) \prod_{p|k} \left(1 - \frac{1}{F(p)}\right) \\ &= F(m) \sum_{d|k} \frac{\mu(d)}{F(d)} \\ &= F(m)g(k). \end{aligned}$$

Thus, in general, we have

$$\sum_{\substack{d|m \\ k|d}} \mu^2(d)f(d) = F(m)\mu^2(k)g(k).$$

The special case is obtained by taking $F(m) = m/\varphi(m)$, so that if k is square-free, then

$$g(k) = \sum_{d|k} \frac{\mu(d)\varphi(d)}{d} = \prod_{p|k} \left(1 - \frac{\varphi(p)}{p}\right) = \prod_{p|k} \frac{1}{p} = \frac{1}{k}.$$

This completes the proof of Lemma 1. \square

Lemma 4 *Let m be a positive integer and let β be any real number. Then*

$$\sum_{\substack{n=1 \\ (n,m)=1}}^{\infty} \frac{e^{2\pi i\beta n}}{n^s} = \sum_{d|m} \frac{\mu(d)}{d^s} \Phi(s, \beta d).$$

Proof. Let x be any complex number with $|x| \leq 1$. Then

$$\begin{aligned} \sum_{\substack{n=1 \\ (n,m)=1}}^{\infty} \frac{x^n}{n^s} &= \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{d|(n,m)} \mu(d) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{\substack{d|n \\ d|m}} \mu(d) = \sum_{d|m} \mu(d) \sum_{k=1}^{\infty} \frac{x^{kd}}{(kd)^s} \\ &= \sum_{d|m} \frac{\mu(d)}{d^s} \sum_{k=1}^{\infty} \frac{x^{kd}}{k^s}. \end{aligned}$$

Replacing x by $e^{2\pi i\beta}$ completes the proof. \square

Proof of Proposition 1. First, recall (see eg. [4, p. 238]) that Ramanujan's sum has the explicit representation

$$c_m(k) = \varphi(m) \frac{\mu(m/(m,k))}{\varphi(m/(m,k))}.$$

Hence, we have

$$\begin{aligned}
& \frac{1}{m} \sum_{b \bmod m} \Phi(s, b/m) c_m(a-b) \\
&= \frac{1}{m} \sum_{b \bmod m} \Phi(s, b/m) \varphi(m) \frac{\mu(m/(m, a-b))}{\varphi(m/(m, a-b))} \\
&= \frac{\varphi(m)}{m} \sum_{d|m} \sum_{\substack{b \bmod m \\ (a-b, m)=m/d}} \Phi(s, b/m) \frac{\mu(d)}{\varphi(d)} \\
&= \frac{\varphi(m)}{m} \sum_{d|m} \frac{\mu(d)}{\varphi(d)} \sum_{\substack{\nu \bmod d \\ (\nu, d)=1}} \Phi\left(s, \frac{a+m\nu/d}{m}\right) \\
&= \frac{\varphi(m)}{m} \sum_{d|m} \frac{\mu(d)}{\varphi(d)} \sum_{n=1}^{\infty} n^{-s} \sum_{\substack{\nu \bmod d \\ (\nu, d)=1}} \zeta_m^{(a+m\nu/d)n} \\
&= \frac{\varphi(m)}{m} \sum_{d|m} \frac{\mu(d)}{\varphi(d)} \sum_{n=1}^{\infty} n^{-s} \zeta_m^{an} c_d(n) \\
&= \frac{\varphi(m)}{m} \sum_{d|m} \frac{\mu(d)}{\varphi(d)} \sum_{n=1}^{\infty} n^{-s} \zeta_m^{an} \varphi(d) \frac{\mu(d/(n, d))}{\varphi(d/(n, d))} \\
&= \frac{\varphi(m)}{m} \sum_{d|m} \mu(d) \sum_{f|d} \frac{\mu(f)}{\varphi(f)} \sum_{\substack{n=1 \\ (n, f)=1}}^{\infty} (nd/f)^{-s} \zeta_{fm/d}^{an} \\
&= \frac{\varphi(m)}{m} \sum_{d|m} \mu(d) \sum_{f|d} \frac{\mu(f)}{\varphi(f)} \left(\frac{d}{f}\right)^{-s} \sum_{\substack{n=1 \\ (n, f)=1}}^{\infty} n^{-s} \zeta_{fm/d}^{an}. \tag{4}
\end{aligned}$$

But by Lemma 4 the final expression in (4) can be rewritten as

$$\frac{\varphi(m)}{m} \sum_{d|m} \mu(d) \sum_{f|d} \frac{\mu(f)}{\varphi(f)} \left(\frac{d}{f}\right)^{-s} \sum_{\delta|f} \delta^{-s} \mu(\delta) \Phi\left(s, \frac{ad\delta}{fm}\right). \tag{5}$$

Now transform (5) by changing the variable f to d/f , then letting $k = f\delta$ (noticing that the only non-zero terms occur when d is square-free), then observing that $\sum_{f|k} \varphi(f) = k$, and finally replacing d by kd . Thus, from (4) and (5),

$$\begin{aligned}
& \frac{1}{m} \sum_{b \bmod m} \Phi(s, b/m) c_m(a-b) \\
&= \frac{\varphi(m)}{m} \sum_{d|m} \mu(d) \sum_{f|d} \frac{\mu(d/f)}{\varphi(d/f)} f^{-s} \sum_{\delta|(d/f)} \delta^{-s} \mu(\delta) \Phi(s, af\delta/m) \\
&= \frac{\varphi(m)}{m} \sum_{d|m} \sum_{k|d} \frac{\mu^2(d)}{\varphi(d)} k^{-s} \mu(k) \Phi(s, ak/m) \sum_{f|k} \varphi(f)
\end{aligned}$$

$$= \frac{\varphi(m)}{m} \sum_{k|m} k^{1-s} \mu(k) \Phi(s, ak/m) \sum_{\substack{d|m \\ k|d}} \frac{\mu^2(d)}{\varphi(d)}. \quad (6)$$

By applying Lemma 3 to (6) and then replacing m/k by d , we find that

$$\begin{aligned} \frac{1}{m} \sum_{b \bmod m} \Phi(s, b/m) c_m(a-b) &= \sum_{k|m} k^{-s} \mu(k) \Phi(s, ak/m) \\ &= \frac{1}{m^s} \sum_{d|m} d^s \mu(m/d) \Phi(s, a/d). \end{aligned} \quad (7)$$

Hence by (7) and Lemma 2, we see that

$$\begin{aligned} \frac{1}{m^s} \sum_{d|m} d^s \mu(m/d) \Phi(s, a/d) &= \frac{1}{m} \sum_{b \bmod m} \Phi(s, b/m) c_m(a-b) \\ &= \frac{1}{\varphi(m)} \sum_{\chi \in \hat{U}_m} \chi(a) \tau(\overline{\chi}) L(s, \chi). \end{aligned}$$

An application of Möbius inversion now completes the proof. \square

3 Main Results

We are now in a position to derive our class number formula. To this end, for $|x| \leq 2$ and $2 \leq k \in \mathbb{Z}$, put

$$s(k, x) := \sum_{n=1}^{\infty} \frac{x^{2n}}{\binom{2n}{n} n^k}.$$

Let $0 < \theta < \pi$ and $x = 2 \sin \theta/2$. Then [3, p. 61 (2)] $2s(2, x) = \theta^2$ and by formula (2.7) of [5],

$$\begin{aligned} \theta^2 \log(2 \sin \theta/2) &= 2\zeta(3) + \sum_{n=1}^{\infty} \frac{(2 \sin \theta/2)^{2n}}{\binom{2n}{n} n^3} \\ &\quad - 2\theta \operatorname{Im} \Phi(2, \theta/2\pi) - 2\operatorname{Re} \Phi(3, \theta/2\pi), \end{aligned} \quad (8)$$

where Re and Im denote the real and imaginary parts of a complex number, respectively. Now substitute $\theta = 2\pi a/m$ with $(a, m) = 1$ and $0 < a < m/2$ in (8) to obtain

$$\begin{aligned} \log \left(2 \sin \frac{\pi a}{m} \right) &= \frac{m^2}{2\pi^2 a^2} \zeta(3) + \frac{m^2}{4\pi^2 a^2} \sum_{n=1}^{\infty} \frac{(2 \sin \pi a/m)^{2n}}{\binom{2n}{n} n^3} \\ &\quad - \frac{m}{\pi a} \operatorname{Im} \Phi \left(2, \frac{a}{m} \right) - \frac{m^2}{2\pi^2 a^2} \operatorname{Re} \Phi \left(3, \frac{a}{m} \right). \end{aligned} \quad (9)$$

In our main result, character sums of consecutive integer powers arise, and it is convenient to fix some notation for these.

Definition 1 Let m be a positive integer. If χ is a Dirichlet character of modulus m and j is any integer, put

$$\mathcal{B}_j(\chi) := \sum_{0 < a < m/2} a^j \chi(a). \quad (10)$$

We now state and prove our main result.

Theorem 1 Let m be a positive integer, let ψ be an even primitive character of modulus m , and let \mathcal{B}_j be as in (10). Then

$$\begin{aligned} L(1, \psi) &= \frac{2\tau(\psi)}{\pi i m^2} \sum_{d|m} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \in \hat{U}_d \\ \chi \text{ odd}}} \mathcal{B}_{-1}(\chi \bar{\psi}) \tau(\bar{\chi}) L(2, \chi) \\ &+ \frac{\tau(\psi)}{\pi^2 m^2} \sum_{d|m} \frac{d^3}{\varphi(d)} \sum_{\substack{\chi \in \hat{U}_d \\ \chi \text{ even}}} \mathcal{B}_{-2}(\chi \bar{\psi}) \tau(\bar{\chi}) L(3, \chi) - \frac{m\tau(\psi)}{\pi^2} \mathcal{B}_{-2}(\bar{\psi}) \zeta(3) \\ &- \frac{m\tau(\psi)}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^3} \sum_{0 < a < m/2} \frac{\bar{\psi}(a)}{a^2} \left(2 \sin \frac{\pi a}{m}\right)^{2n}. \end{aligned}$$

Proof. We start with (9) and write $\text{Im } \Phi(2, a/m)$ and $\text{Re } \Phi(3, a/m)$ in terms of L -series via Proposition 1; but first observe that

$$\begin{aligned} &\text{Im} \left(\sum_{\chi \bmod d} \chi(a) \tau(\bar{\chi}) L(2, \chi) \right) \\ &= \frac{1}{2i} \sum_{\chi \in \hat{U}_d} \left(\chi(a) \tau(\bar{\chi}) L(2, \chi) - \overline{\chi(a) \tau(\bar{\chi}) L(2, \chi)} \right) \\ &= \frac{1}{2i} \sum_{\chi \in \hat{U}_d} \left(\chi(a) \tau(\bar{\chi}) L(2, \chi) - \chi(-1) \bar{\chi}(a) \tau(\chi) L(2, \bar{\chi}) \right), \end{aligned}$$

since $\overline{\tau(\bar{\chi})} = \chi(-1) \tau(\bar{\chi})$. Now split the sum over the two terms and in the second sum replace χ by $\bar{\chi}$. The even characters cancel and we obtain

$$\text{Im} \left(\sum_{\chi \in \hat{U}_d} \chi(a) \tau(\bar{\chi}) L(2, \chi) \right) = \frac{1}{i} \sum_{\substack{\chi \in \hat{U}_d \\ \chi(-1) = -1}} \chi(a) \tau(\bar{\chi}) L(2, \chi).$$

Similarly, we see that

$$\text{Re} \left(\sum_{\chi \in \hat{U}_d} \chi(a) \tau(\bar{\chi}) L(3, \chi) \right) = \sum_{\substack{\chi \in \hat{U}_d \\ \chi(-1) = 1}} \chi(a) \tau(\bar{\chi}) L(3, \chi).$$

Thus by (9) and Proposition 1,

$$\begin{aligned}
\log \left(2 \sin \frac{\pi a}{m} \right) &= \frac{m^2}{2\pi^2 a^2} \zeta(3) + \frac{m^2}{4\pi^2 a^2} \sum_{n=1}^{\infty} \frac{(2 \sin \pi a/m)^{2n}}{\binom{2n}{n} n^3} \\
&- \frac{1}{\pi i m a} \sum_{d|m} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \in \tilde{U}_d \\ \chi \text{ odd}}} \chi(a) \tau(\overline{\chi}) L(2, \chi) \\
&- \frac{1}{2\pi^2 m a^2} \sum_{d|m} \frac{d^3}{\varphi(d)} \sum_{\substack{\chi \in \tilde{U}_d \\ \chi \text{ even}}} \chi(a) \tau(\overline{\chi}) L(3, \chi). \quad (11)
\end{aligned}$$

Next, recall (see eg. [2, p. 336]) that if ψ is an even primitive character of modulus m , then

$$\begin{aligned}
L(1, \psi) &= -\frac{\tau(\psi)}{m} \sum_{a=1}^{m-1} \overline{\psi}(a) \log \left(2 \sin \frac{\pi a}{m} \right) \\
&= -\frac{2\tau(\psi)}{m} \sum_{0 < a < m/2} \overline{\psi}(a) \log \left(2 \sin \frac{\pi a}{m} \right). \quad (12)
\end{aligned}$$

Substituting (12) into (11) completes the proof. \square

Let D be a (positive fundamental) discriminant of a real quadratic number field. Let $h(D)$ denote its class number, $\varepsilon = \varepsilon_D$ its fundamental unit > 1 , and $\chi_D = (D/\cdot)$, the Kronecker symbol, i.e. the Dirichlet character associated with the quadratic field of discriminant D . Then by Dirichlet (see eg. [2, p. 343]), we know that

$$2h(D) \log \varepsilon_D = \sqrt{D} L(1, \chi_D).$$

Hence by Theorem 1, using the fact that $\tau(\chi_D) = \sqrt{D}$, we obtain the following class number formula.

Corollary 1 (Class Number Formula)

$$\begin{aligned}
h(D) \log \varepsilon_D &= \frac{1}{\pi D i} \sum_{d|D} \frac{d^2}{\varphi(d)} \sum_{\substack{\chi \in \tilde{U}_d \\ \chi \text{ odd}}} \mathcal{B}_{-1}(\chi \chi_D) \tau(\overline{\chi}) L(2, \chi) \\
&+ \frac{1}{2\pi^2 D} \sum_{d|D} \frac{d^3}{\varphi(d)} \sum_{\substack{\chi \in \tilde{U}_d \\ \chi \text{ even}}} \mathcal{B}_{-2}(\chi \chi_D) \tau(\overline{\chi}) L(3, \chi) \\
&- \frac{D^2}{2\pi^2} \mathcal{B}_{-2}(\chi_D) \zeta(3) \\
&- \frac{D^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^3} \sum_{0 < a < D/2} \frac{\chi_D(a)}{a^2} \left(2 \sin \frac{\pi a}{D} \right)^{2n}.
\end{aligned}$$

3.1 A Computation

As an amusing conclusion, we now show how to use our class number formula to compute $h(5)$, the class number of the quadratic field $\mathbb{Q}(\sqrt{5})$. Since the discriminant $D = 5$, the only relevant moduli of characters are $m = 1$ and $m = 5$. For $m = 1$, the unique character is the even constant character 1. For $m = 5$, we have four characters determined by the homomorphisms from $(\mathbb{Z}/5\mathbb{Z})^\times$ into \mathbb{C}^\times , namely χ_ν for $\nu = 0, 1, 2, 3$ determined by $\chi_\nu(2) = i^\nu$. Notice that $\overline{\chi_1} = \chi_3$ and that $\chi_2 = (5/\cdot) = \chi_5$, the Kronecker character modulo 5.

By Corollary 1, we have $h(5) = (A + B + C + S)/\log \varepsilon_5$, where

$$\begin{aligned} A &= \frac{5}{4\pi i} (\mathcal{B}_{-1}(\chi_3)\tau(\chi_3)L(2, \chi_1) + \mathcal{B}_{-1}(\chi_1)\tau(\chi_1)L(2, \chi_3)), \\ B &= \frac{1}{10\pi^2} \left(\mathcal{B}_{-2}(\chi_5)\tau(1)\zeta(3) \right. \\ &\quad \left. + \frac{125}{4} (\mathcal{B}_{-2}(\chi_5)\tau(\chi_0)L(3, \chi_0) + \mathcal{B}_{-2}(\chi_0)\tau(\chi_5)L(3, \chi_5)) \right) \\ C &= -\frac{25}{2\pi^2} \mathcal{B}_{-2}(\chi_5)\zeta(3) \\ S &= -\frac{25}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^3} (\chi_5(1)(2 \sin \pi/5)^{2n} + \frac{1}{4} \chi_5(2)(2 \sin 2\pi/5)^{2n}), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_{-1}(\chi_1) &= \chi_1(1) + \frac{1}{2}\chi_1(2) = 1 + \frac{1}{2}i \\ \mathcal{B}_{-1}(\chi_3) &= 1 - \frac{1}{2}i \\ \mathcal{B}_{-2}(\chi_0) &= 1 + \frac{1}{4} = \frac{5}{4} \\ \mathcal{B}_{-2}(\chi_5) &= 1 - \frac{1}{4} = \frac{3}{4} \\ \tau(1) &= 1 \\ \tau(\chi_0) &= -1 \\ \tau(\chi_5) &= \sqrt{5} \\ \tau(\chi_1) &= \zeta_5 + i\zeta_5^2 - i\zeta_5^3 - \zeta_5^4 = \left(i + \frac{1 - \sqrt{5}}{2} \right) \sqrt{\frac{5 + \sqrt{5}}{2}} \\ \tau(\chi_3) &= \zeta_5 - i\zeta_5^2 + i\zeta_5^3 - \zeta_5^4 = \left(i + \frac{\sqrt{5} - 1}{2} \right) \sqrt{\frac{5 + \sqrt{5}}{2}} \\ L(3, \chi_0) &= (1 - 5^{-3}) \zeta(3) = \frac{124}{125} \zeta(3). \end{aligned}$$

In order to evaluate $L(s, \chi_\nu)$ for $\nu = 0, 1, 2, 3$ and $s = 2, 3$ we write

$$L(s, \chi_\nu) = 5^{-s} \sum_{r=1}^4 \chi_\nu(r) \zeta(s, r/5),$$

where $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ is the Hurwitz zeta function. Hence to evaluate these L -series, it suffices to evaluate the Hurwitz zeta functions. The following table gives the appropriate approximations.

r	$\zeta(2, r/5)$	$\zeta(3, r/5)$
1	26.26737720...	125.73901805...
2	7.27535659...	16.1195643...
3	3.63620967...	4.98141576...
4	2.29947413...	2.21505785...

Thus, we find that

$$\begin{aligned} L(2, \chi_1) &= 0.95871612... + (0.14556587...)i \\ L(2, \chi_3) &= 0.95871612... - (0.14556587...)i \\ L(3, \chi_5) &= 0.85482476... \end{aligned}$$

Furthermore, $\zeta(3) = 1.20205690...$, so that

$$\begin{aligned} A &= 1.24907310... \\ B &= 0.48248793... \\ C &= -1.14181713... \\ S &= -0.10853146... \end{aligned}$$

Finally, the fundamental unit

$$\varepsilon_5 = \frac{1 + \sqrt{5}}{2},$$

(which generally can be computed efficiently by continued fractions).

From all of this we obtain $h(5) = 1.0000000... + (0.0000000...)i$, whence $h(5) = 1$.

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